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Wave Propagation in Inhomogenous Slab Waveguides Embedded in Homogenous Media

AKIRA YATA AND HIROYOSHI IKUNO

Abstract —**Wave propagation in inhomogeneous slab waveguides embedded in homogeneous media is analyzed by using the uniform asymptotic technique. This technique accurately evaluates the effect of the refractive-index profiles with various core and cladding structures on the guided modes. We calculate the guided modes of waveguides with asymmetric claddings in the cases of a near-parabolic profile core and a quasi-Gaussian profile core. The results show that the third-order asymptotic solution is accurate for all the guided modes in the case of the near-parabolic profile core and for modes far from cutoff in the quasi-Gaussian core case. The dispersion relation indicates that modes guided in strongly asymmetric profiles have almost the same propagation constants as odd-order modes of propagation in the symmetric structure.**

I. INTRODUCTION

ADVANCES OF integrated optics produce optical channel waveguides and directional couplers with a great variety of refractive-index distributions. In such waveguides, propagation characteristics of the guided modes are strongly influenced by the inhomogeneous core and the uniform cladding. A number of numerical methods are used for evaluating the effect of the structure on guided modes [1]–[4]. A more efficient analytic method is desired for studying the wave propagation in a typical Gaussian slab waveguide structure.

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The uniform asymptotic method developed in a companion paper [5] is a useful approximate technique for analyzing the guided modes of a considerably general class of refractive-index profiles. We derive an expression for calculating the third-order correction to the WKB solution of the guided modes of the uncladded waveguide in which the core variety is described by an even polynomial refractive-index profile. When considering the cladded waveguides, we need to replace the *integer mode index* of the ideal waveguide eigenvalues and modal fields with the *non-integer mode index*. The unknown non-integer mode index is determined by solving the characteristic equation which is derived from the boundary conditions at the core-cladding interfaces. In this formulation, the cladding effect is automatically included in the non-integer mode index [6], [7]. The method presented here is a uniform asymptotic approach in the sense that the modal field distributions of the waveguide with various refractive-index profiles are obtained in all orders for all points across the cross section.

As an actual example, we calculate the third-order approximate solution of the non-integer mode index in the case of the asymmetric refractive-index distribution. The convergence and the accuracy of the solutions are checked numerically in two cases; the near-parabolic profile core and the quasi-Gaussian profile core. Lastly, we examine fundamental properties of a strongly asymmetric waveguide by comparing it with the symmetric profile case.

II. Characteristic Equation

We consider the two-dimensional waveguides with the asymmetric and uniform cladding as shown in Fig. 1. The refractive-index distribution is represented as

$$n(x)=\begin{cases} n_0(1-2\Delta_1)^{1/2}, & x<a \\ n_0(1-h(x))^{1/2}, & a\leqslant x\leqslant a+d \\ n_0(1-2\Delta_2)^{1/2}, & a+d<x \end{cases} \quad (1)$$

where n_0 is the refractive-index at $x=0$, d is the width of the inhomogeneous part of the waveguide, and a is a constant. We assume, for convenience, $0\leqslant\Delta<\Delta_1\leqslant\Delta_2$, where the minimum value of $h(x)$ in the interval $a\leqslant x\leqslant a+d$ is 2Δ. Here we use a Cartesian coordinate system (x,y,z). We describe a variety of cores including the Gaussian distribution by a monotonically increasing function $h(x)$ such that [5]

$$h(x)=(gx)^2-a_2(gx)^4+a_3(gx)^6+\cdots+(-1)^{M+1}a_M(gx)^{2M} \quad (2)$$

where a_M's are constants and g is the grading parameter. Refractive-indices $n_1(=n_0(1-2\Delta_1)^{1/2})$ and $n_2(=n_0(1-2\Delta_2)^{1/2})$ are parameters to represent the various cladding structures.

The guided waves propagate along the z-axis in such a guiding medium according to $\exp(j(\omega t-\beta_\mu z))$, where β_μ is a propagation constant. The effect of homogeneous claddings on the guided modes is included in an unknown parameter μ only [6], [7]. The problem of determining the guided modes in the cladded waveguides is to seek the transverse mode functions $\Phi_\mu(x)$ and the eigenvalues b_μ satisfying the following equation:

$$\Phi_\mu''(x)+k^2Q(x)\Phi_\mu(x)=0$$

$$Q(x)=\begin{cases} b_\mu-2\Delta_1, & x<a \\ b_\mu-\chi(x), & a\leqslant x\leqslant a+d \\ b_\mu-2\Delta_2, & a+d<x \end{cases}$$

$$\beta_\mu=k(1-b_\mu)^{1/2}, \qquad k=(2\pi/\lambda)n_0 \quad (3)$$

where the prime denotes the derivative with respect to x, λ is the wavelength in vacuum, and

$$\chi(x)=h(x)+\begin{cases} 0, & \text{for TE-modes} \\ (1/k^2)(1-h(x))^{1/2}\left((1-h(x))^{-1/2}\right)'', & \text{for TM-modes.} \end{cases}$$

Now we derive the characteristic equation for determining the unknown μ of the guided modes. First, we solve (3). In the companion paper [5], (3) is solved by the uniform asymptotic method in the core region. Replacing the *integer mode index m* in those solutions by the *non-integer mode index* μ, we have the Nth-order approximate eigenvalues and field distributions. We can write the Nth-order approximate modal fields by the linearly combined form of two independent field solutions $U_1(x)$ and $U_2(x)$ of (3) as

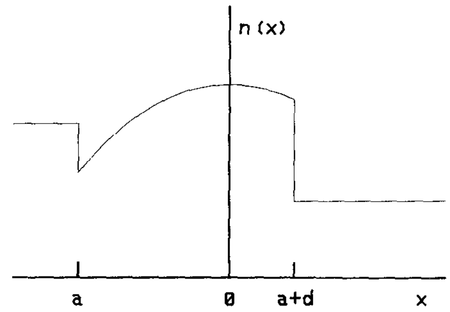

Fig. 1. Refractive-index profile.

$$\Phi_\mu^{(N)}(x)=C_1U_1(x)+C_2U_2(x), \qquad \text{for } a\leqslant x\leqslant a+d \quad (4)$$

where C_1 and C_2 are unknown constants. Referring to [6] and [5, eq. (14)], we can express $U_1(x)$ and $U_2(x)$ of (4) as follows:

$$U_1(x)=\left(dw_{N+1}(w_1)/dw_1\right)^{-1/2}D_\mu\left(-\sqrt{4\mu+2}\,w_{N+1}(w_1)\right) \quad (5a)$$

$$U_2(x)=\left(dw_{N+1}(w_1)/dw_1\right)^{-1/2}D_\mu\left(\sqrt{4\mu+2}\,w_{N+1}(w_1)\right) \quad (5b)$$

where w_{N+1} and w_1 are the same symbols as those in [5]. It is noted that the field function $U_1(x)$ tends to 0 at $x=-\infty$ and diverges at $x=\infty$, and vice versa for $U_2(x)$. In the cladding, the transverse mode functions are expressed as

$$\Phi_\mu(x)=C_3\exp\left(G_1(x-a)\right),$$
$$G_1=k(2\Delta_1-b_\mu)^{1/2}, \qquad \text{for } x<a \quad (6)$$

$$\Phi_\mu(x)=C_4\exp\left(G_2(a+d-x)\right),$$
$$G_2=k(2\Delta_2-b_\mu)^{1/2}, \qquad \text{for } a+d<x \quad (7)$$

where C_3 and C_4 are unknown constants. Matching the tangential field components at the core-cladding interfaces $x=a$ and $x=a+d$ gives the following characteristic equation:

$$\left(F_1U_1(a)-U_1'(a)\right)/\left(F_1U_2(a)-U_2'(a)\right)-\left(F_2U_1(a+d)+U_1'(a+d)\right)/\left(F_2U_2(a+d)+U_2'(a+d)\right)=0 \quad (8)$$

with

$$F_1=\begin{cases} G_1, & \text{for TE-modes} \\ G_1(1-h(a))/(1-2\Delta_1)+(1/2)h'(a)/(1-h(a)), & \text{for TM-modes} \end{cases}$$

$$F_2=\begin{cases} G_2, & \text{for TE-modes} \\ G_2(1-h(a+d))/(1-2\Delta_2)-(1/2)h'(a+d)/(1-h(a+d)), & \text{for TM-modes.} \end{cases}$$

Equation (8) determines the unknown parameter μ. Simultaneously, we have the following relations:

$$C_2/C_1 = -(F_1U_1(a) - U_1'(a))/(F_1U_2(a) - U_2'(a))$$
$$C_3/C_1 = F_3(U_1(a) + (C_2/C_1)U_2(a))$$
$$C_4/C_1 = F_4(U_1(a+d) + (C_2/C_1)U_2(a+d)) \quad (9)$$

with

$$F_3 = \begin{cases} 1, & \text{for TE-modes} \\ ((1-h(a))/(1-2\Delta_1))^{1/2}, & \text{for TM-modes} \end{cases}$$

$$F_4 = \begin{cases} 1, & \text{for TE-modes} \\ ((1-h(a+d))/(1-2\Delta_2))^{1/2}, & \text{for TM-modes} \end{cases}$$

Equation (9) serves us to draw the field pattern in the whole region. In the symmetric profile case ($a = -d/2$, $n_1 = n_2$), (8) is reduced to the simplified form

$$F_1(U_1(a) + U_2(a)) - (U_1'(a) + U_2'(a)) = 0, \quad \text{for even-order modes} \quad (10a)$$

$$F_1(U_1(a) - U_2(a)) - (U_1'(a) - U_2'(a)) = 0, \quad \text{for odd-order modes.} \quad (10b)$$

By solving (8), we can evaluate the propagation characteristics of the guided modes of the dielectric slab waveguides with inhomogeneous core and uniform claddings.

III. Numerical Calculation

First, we check the convergence of the uniform asymptotic solution presented here by calculating the cladding effect μ. We introduce several parameters. By setting $a_M = (1/M!)(1/K^{M-1})$ and $g = \sqrt{2\Delta_1}/D$ in (2), we can realize the near-parabolic profile core when $K = 1$ and the quasi-Gaussian profile core when $K = 2\Delta_1$ [5]. The parameter D is the Gaussian profile core width. We also consider the cladding structure of the asymmetric refractive-index distribution such that

$$h(a) = 2\Delta_1 \quad (11a)$$

$$h(a+d) = 0. \quad (11b)$$

Equations (11) and (2) relate the core width D of the Gaussian profile to the core width d of waveguides as shown in Fig. 1, and (11b) indicates that there is one abrupt transition of refractive-index distribution at $x = 0$. In this paper, the normalized frequency V and the normalized phase parameter B are defined by

$$V = kD\sqrt{2\Delta_1} \quad (12)$$

$$B = 1 - b_\mu/2\Delta_1. \quad (13)$$

We evaluate the cladding effect on TE$_m$-modes and TM$_m$-modes of the asymmetric profile waveguide with the cladding given by (11). In the limit of infinite asymmetry, all the modes of propagation in the asymmetric structure waveguide correspond to odd-order modes of the symmetric profile [8]. So the non-integer mode index μ is expressed in the form

$$\mu = (2m+1) + \Delta\mu, \quad m = 0, 1, 2, \cdots \quad (14)$$

where $\Delta\mu$ is the cladding effect. We calculate $\Delta\mu^{(N)}$ under the condition $b_\mu^{(N)}/2\Delta_1 = 0.67$ ($B = 0.33$), where the third-order asymptotic solution of (3) is used. Fig. 2 shows that the solutions $\Delta\mu^{(3)}$ are converging in cases of all the guided modes. We examine the accuracy of the solutions. Table I shows that the significant figures of μ in the case of the near-parabolic profile core are more than ten at cutoff ($B = 0$) and the quasi-Gaussian profile core case has significant figures more than four at the frequency far from cutoff ($B = 0.33$). We conclude that the third-order asymptotic eigenvalues and modal fields are sufficient to evaluate the cladding effect on the guided modes of waveguides with the near-parabolic profile core, but insufficient

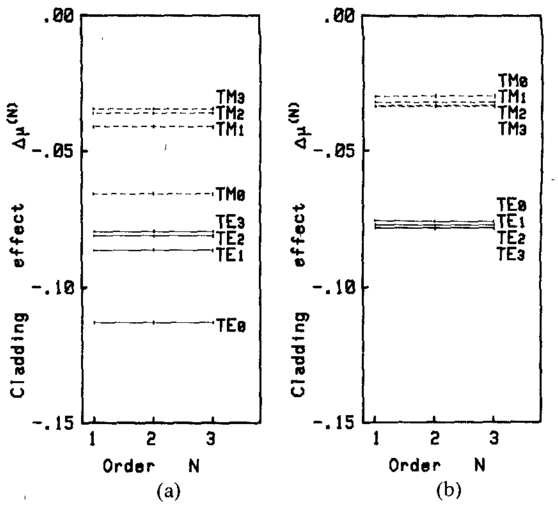

Fig. 2. Convergence of $\Delta\mu^{(N)}$ of the asymmetric waveguide with $n_0 = 1.53$, $n_1 = 1.52$, and $n_2 = 1.0$ when $B = 0.33$. (a) Near-parabolic profile core case. (b) Quasi-Gaussian profile core case.

TABLE I
Calculated Mode Index μ in the Cases of the Near-Parabolic Profile Core (Upper) and the Quasi-Gaussian Profile Core (Lower)

mode	B=0 (cutoff)	B=.33
TE$_0$	.5297534614	.8870422837
	.7	.9238
TM$_0$	.5834480353	.9341287191
	.7	.9703
TE$_1$	2.551934570	2.913604595
	2.7	2.9227
TM$_1$	2.605035745	2.958948417
	2.8	2.9678
TE$_2$	4.557413963	4.918862137
	4.8	4.9216
TM$_2$	4.610872026	4.963979194
	4.8	4.9666
TE$_3$	6.560061210	6.920382095
	6.8	6.9214
TM$_3$	6.613762304	6.965434190
	6.9	6.9663

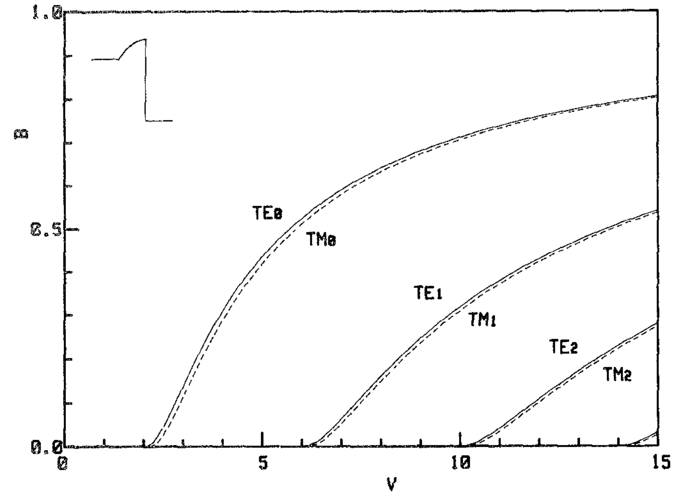

Fig. 3. Phase parameter of lower order modes in the strongly asymmetric waveguide with the near-parabolic profile core.

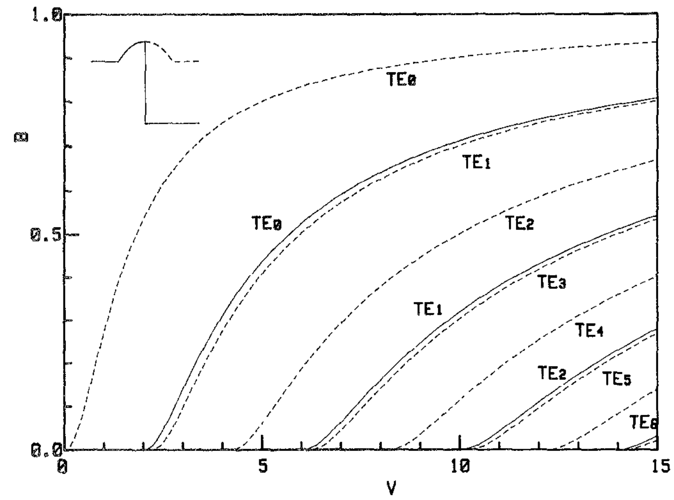

Fig. 4. Comparison of phase parameters in the strongly asymmetric waveguide and the symmetric waveguide with $n_0 = 1.53$ and $n_1 = n_2 = 1.52$. The solid curves show the asymmetric profile case and the dashed curves show the symmetric case.

for near cutoff modes in the quasi-Gaussian profile core case.

Next, we analyze wave propagations of the strongly asymmetric profile in the near-parabolic case by using the third-order asymptotic solution. Fig. 3 gives the phase parameter B as a function of the normalized frequency V for TE-modes and TM-modes of lower orders. This figure shows that for given V-values the parameters B are always smaller for TM_m-modes than for the TE_m-modes of corresponding m. This is a property similar to that of asymmetric step-index waveguides [8]. TM-modes have almost the same properties as TE-modes. So we calculate only the case of the TE-modes in the following discussion. Fig. 4 supports that all odd-order modes of the symmetric profile case correspond to modes of propagation in the strongly asymmetric structure waveguides [8]. This fact indicates that the fundamental modes in the asymmetric waveguides have finite cutoff frequencies. This technique can not analyze the wave propagation of the lowest order modes at very low frequency ($V < 0.2$) in the symmetric case as shown in this figure because the method adopted here is an asymptotic technique.

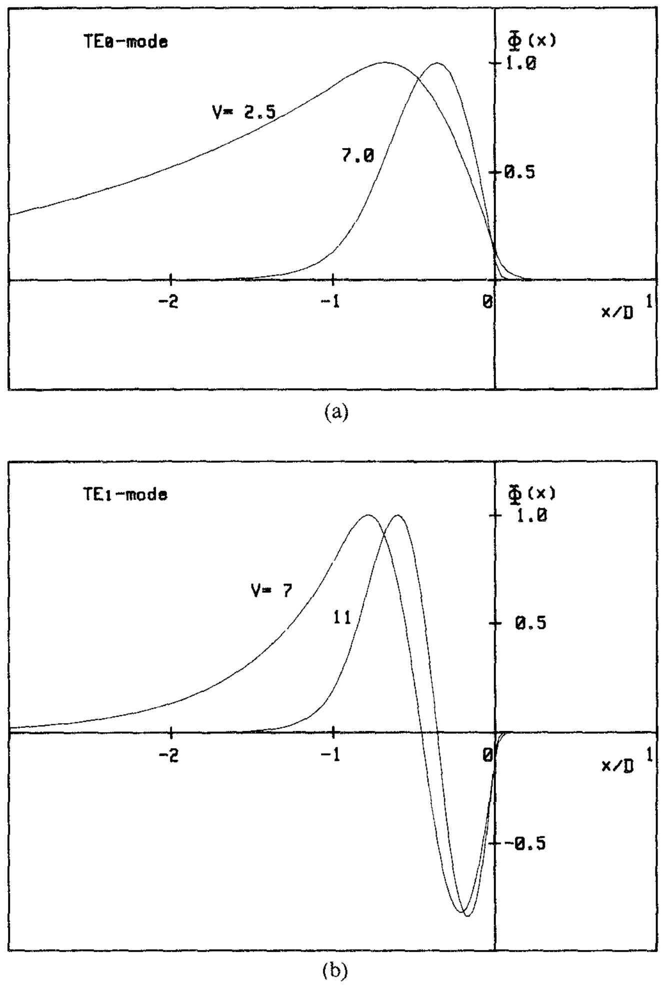

Fig. 5. Transverse field distribution in the strongly asymmetric waveguide. (a) TE_0-mode. (b) TE_1-mode.

Finally, we compute field distributions in the asymmetric profile core case. Fig. 5(a) shows the field patterns of the TE_0-mode, near the cutoff case ($V = 2.5$), and the case far from cutoff ($V = 7.0$). The field pattern (when $V = 2.5$) indicates that the energy leaks to the left side of the cladding region. These field patterns fit almost into a half of the TE_1-mode field pattern in the case of the symmetric profile [8]. The pattern of TE_1-mode is shown in Fig. 5(b) in two cases $V = 7$ and $V = 11$. It is noted that field patterns as shown in this figure are illustrated accurately in the whole region to be considered because the technique adopted here is the uniform asymptotic method.

IV. Conclusion

We present the uniform asymptotic technique for handling the wave propagation of inhomogeneous slab waveguides embedded in homogeneous media. By the technique presented here, the cladding effect on the guided modes is evaluated accurately. The non-integer mode index involves the cladding effect on the guided modes in this formulation. The convergence and the accuracy of solution are examined for the asymmetric waveguides with the near-parabolic profile core and the quasi-Gaussian profile core numerically. It is found that the third-order approximate solution is accurate for all the guided modes in the case of the near-parabolic profile core and for modes far from cutoff in the quasi-Gaussian profile core case. The numerical results show that propagation modes in the strongly asymmetric profile case correspond almost to all odd-order modes of propagation in the symmetric profile waveguide; the fundamental modes in the asymmetric structure waveguides have finite cutoff frequencies.

A still higher order solution for eigenvalues and modal fields is needed to evaluate the propagation characteristics of the Gaussian profile core waveguide. This will be done in the near future. We can also analyze the leaky modes by the technique presented here.

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